

ASSOCIATIVE ALGEBRAS UNDER MULTI-COMMUTATORS

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ABSTRACT. For an associative algebra A a skew-symmetric (symmetric) sum of $n!$ products of n elements of A in all possible order is called Lie (Jordan) n -commutator. We consider A as n -ary algebra under n -commutator. We construct n -ary skew-symmetric and symmetric generalizations of Jordan identity. We prove that any associative algebra under Jordan n -commutator satisfies a symmetric generalization of Jordan identity. We prove that in case of odd n any associative algebra under Lie n -commutator satisfies a skew-symmetric generalization of Jordan identity. In case of even n Lie n -commutator satisfies the homotopical n -Lie identity.

Well known that an associative algebra A under Lie commutator is Lie. In other words, a vector space A under commutator $[a, b] = ab - ba$ has skew-symmetric multiplication $[\cdot, \cdot] : \wedge^2 A \rightarrow A$, that satisfies the identity, called Jacobi identity

$$[a_1, [a_2, a_3]] - [a_2, [a_1, a_3]] + [a_3, [a_1, a_2]] = 0.$$

Well known also, that an associative algebra A under Jordan commutator $\{a, b\} = ab + ba$ is Jordan. In other words, Jordan commutator is symmetric multiplication $\{\cdot, \cdot\} : S^2 A \rightarrow A$, that satisfies the identity of degree 4, called Jordan identity

$$\begin{aligned} & \{a_1, \{a_0, \{a_2, a_3\}\}\} + \{a_2, \{a_0, \{a_1, a_3\}\}\} + \{a_3, \{a_0, \{a_1, a_2\}\}\} \\ & - \{\{a_0, a_1\}, \{a_2, a_3\}\} - \{\{a_0, a_2\}, \{a_1, a_3\}\} - \{\{a_0, a_3\}, \{a_1, a_2\}\} = 0. \end{aligned}$$

In our paper we consider multi-versions of these connections. We answer to a question of A.G. Kurosh who asked about identities of multi-associative algebras under multi-commutator [5]. We show that an associative algebra under skew-symmetric n -commutator satisfies a homotopy identity (generalisation of Jacobi identity) if n is even and one skew-symmetric generalization of Jordan identity if n is odd. We establish that an associative algebra under symmetric n -commutator satisfies symmetric generalization of Jordan identity.

To formulate our results we need to introduce some definitions. Let A be a vector space over a field K . For a multilinear map $\alpha : A \times \cdots \times A \rightarrow A$ we say that $A = (A, \alpha)$ is n -algebra with n -multiplication α . A n -algebra A is said *skew-commutative* if α is skew-symmetric,

$$\alpha(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = \text{sign } \sigma \alpha(a_1, \dots, a_n),$$

for any permutation $\sigma \in \text{Sym}_n$. Similarly, (A, α) is *commutative* n -algebra, if

$$\alpha(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = \alpha(a_1, \dots, a_n),$$

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for any $\sigma \in \text{Sym}_n$.

An absolute free n -algebra (free n -magma) can be defined as algebra of (n -non-commutative, n -non-associative) n -polynomials $K\langle t_1, t_2, \dots \rangle$. Denote by ω a n -multiplication in free n -magma. To construct n -polynomials we have to introduce n -monoms.

By definition, any variable t_i is a n -monom of ω -degree 0. If f_i is a n -monom of ω -degree k_i , and $i = 1, \dots, N$, then $\omega(f_1, \dots, f_N)$ is a n -monom of ω -degree $k_1 + \dots + k_N + 1$. A linear combination of n -monoms is called (n -non-commutative, n -non-associative) n -polynomial. A space of n -polynomials $K\langle t_1, t_2, \dots \rangle$ is defined as a linear space with base generated by n -monoms. A multiplication ω on $K\langle t_1, t_2, \dots \rangle$ is defined in a natural way. If $g_1, \dots, g_N \in K\langle t_1, t_2, \dots \rangle$, then by multilinearity $\omega(g_1, \dots, g_N)$ is a linear combination of n -monoms. We can imagine n -monoms as a rooted tree, where each vertex has n -in edges and 1-out edge. Leaves are labeled by elements of algebra and to inner vertices correspond n -ary products of elements that come by in-edges.

Let $f = f(t_1, \dots, t_k)$ be any n -polynomial of $K\langle t_1, t_2, \dots \rangle$. Let (A, α) be any n -algebra with n -multiplication α . For any k elements $a_1, \dots, a_k \in A$ one can make substitutions $t_i := a_i$ and $\omega := \alpha$ in polynomial f and consider another element $f(a_1, \dots, a_k)$ of A where multiplications are made in terms of multiplication α instead of ω . We say that $f = 0$ is a n -identity on A if $f(a_1, \dots, a_k) = 0$ for any $a_1, \dots, a_k \in A$.

In case of $n = 2$ we obtain usual algebras. In 2-algebras multiplications are usually denoted as $a \circ b$, $a \times b$, $a + b$, etc, instead of $\alpha(a, b)$. The notions of 2-polynomials and 2-polynomial identities coincide with usual notions of polynomials and polynomial identities

Let

$$s_n = s_n(t_1, \dots, t_n) = \sum_{\sigma \in \text{Sym}_n} \text{sign } \sigma t_{\sigma(1)} \cdots t_{\sigma(n)}$$

be a standard (associative, non-commutative) skew-symmetric polynomial. Then any associative algebra A with 2-multiplication ab can be endowed by a structure of n -algebra given by n -multiplication $s_n(a_1, \dots, a_n)$. Call

$$[a_1, \dots, a_n] = s_n(a_1, \dots, a_n)$$

as *Lie n -commutator*. Note that Lie 2-commutator coincides with usual Lie commutator,

$$s_2(a_1, a_2) = a_1 a_2 - a_2 a_1.$$

Let

$$s_n^+ = s_n^+(t_1, \dots, t_n) = \sum_{\sigma \in \text{Sym}_n} t_{\sigma(1)} \cdots t_{\sigma(n)}$$

be a standard (associative, non-commutative) symmetric polynomial. We can endow any associative algebra A with 2-multiplication ab by a structure of n -algebra given by *Jordan n -commutator*

$$\{a_1, \dots, a_n\} = s_n^+(a_1, \dots, a_n).$$

Note that Jordan 2-commutator coincides with usual Jordan commutator,

$$s_2^+(a_1, a_2) = a_1 a_2 + a_2 a_1.$$

In our paper we study n -polynomial identities of the algebra (A, s_n) , and (A, s_n^+) if 2-algebra A is associative. In fact we construct generalizations of Lie

and Jordan identities that hold for total associative algebras under Lie and Jordan n -commutators.

1. Formulations of main results

Let Sym_n be set of all permutations on $[n] = \{1, 2, \dots, n\}$. Let

$$S_{k,l} = \{\sigma \in Sym_{k+l} | \sigma(1) < \dots < \sigma(k), \sigma(k+1) < \dots < \sigma(k+l)\}$$

$$S_{n-1,n-1,n} = \{\sigma \in Sym_{3n-2} | \sigma(1) < \dots < \sigma(n-1), \\ \sigma(n) < \dots < \sigma(2n-2), \quad \sigma(2n-1) < \dots < \sigma(3n-2)\},$$

$S_{n-2,n,n} = \{\sigma \in Sym_{3n-2} | \sigma(1) < \dots < \sigma(n-2), \\ \sigma(n-1) < \dots < \sigma(2n-2), \quad \sigma(2n-1) < \dots < \sigma(3n-2), \quad \sigma(n-1) < \sigma(2n-1)\}.$
are subsets of shuffle-permutations

If n -multiplication $\omega(t_1, \dots, t_n)$ is skew-symmetric, then there exists only one n -monomial of ω -degree 2

$$H(t_1, \dots, t_{2n-1}) = \omega(t_1, \dots, t_{n-1}, \omega(t_n, \dots, t_{2n-1})).$$

Let

$$h(t_1, \dots, t_{2n-1}) = \frac{1}{(n-1)!n!} \omega(t_{[1], \dots, t_{n-1}, \omega(t_n, \dots, t_{(2n-1]})$$

be its skew-symmetrisation by all parameters. Note that,

$$h(t_1, \dots, t_{2n-1}) = \sum_{\sigma \in Sym_{n-1,n}} \text{sign } \sigma \omega(t_{\sigma(1)}, \dots, t_{\sigma(n-1)}, (t_{\sigma(n)}, \dots, t_{\sigma(2n-1)})).$$

For skew-commutative n -algebras there are two n -monomials of ω -degree 3

$$F_1(t_1, \dots, t_{3n-2}) = (t_1, t_2, \dots, t_{n-1}, (t_n, \dots, t_{2n-2}, (t_{2n-1}, \dots, t_{(3n-2)})))$$

and

$$F_2(t_1, \dots, t_{3n-2}) = (t_1, t_2, \dots, t_{n-2}, (t_{n-1}, \dots, t_{2n-2}), (t_{2n-1}, \dots, t_{(3n-2)})))$$

Let us introduce their skew-symmetric sums by all parameters except t_1 ,

$$F_1^{[2]}(t_1, \dots, t_{3n-2}) = \frac{1}{(n-1)!(n-2)!n!} \omega(t_{[2], t_3, \dots, t_n, \omega(t_{[1]}, t_{n+1}, \dots, t_{2n-2}, \omega(t_{2n-1}, \dots, t_{(3n-2)}]))),$$

$$F_2^{[2]}(t_1, \dots, t_{3n-2}) = \frac{1}{(n-2)!(n-1)!n!} \omega(t_{[2], t_3, \dots, t_{n-1}, \omega(t_{[1]}, t_n, \dots, t_{2n-2}), \omega(t_{2n-1}, \dots, t_{(3n-2)}))).$$

Upper index s in $F_l^{[s]}$ corresponds to the ω -place where t_1 is and lower index l corresponds to n -bracketing types of ω -degree 3. We have,

$$F_1^{[2]}(t_1, \dots, t_{3n-2}) = \sum_{\sigma \in Sym_{n-1,n-1,n}, \sigma(n)=1} \text{sign } \sigma (t_{\sigma(1)}, \dots, t_{\sigma(n-1)}, (t_1, t_{\sigma(n+1)}, \dots, t_{\sigma(2n-2)}, (t_{\sigma(2n-1)}, \dots, t_{\sigma(3n-2)}))),$$

$$F_2^{[2]}(t_1, \dots, t_{3n-2}) = \sum_{\sigma \in Sym_{n-2,n,n}, \sigma(n-1)=1} \text{sign } \sigma (t_{\sigma(1)}, \dots, t_{\sigma(n-2)}, (t_1, t_{\sigma(n)}, \dots, t_{\sigma(2n-2)}), (t_{\sigma(2n-1)}, \dots, t_{\sigma(3n-2)}))).$$

Let

$$f_\lambda^{[2]} = F_1^{[2]} + \lambda F_2^{[2]}.$$

These notions have symmetric analogs. We save the same notations as in skew-symmetric case. Just change brackets of the form $[,]$ to $\{ , \}$.

For commutative n -algebras there are two n -monomials of ω -degree 3

$$F_1^+(t_1, \dots, t_{3n-2}) = (t_1, t_2, \dots, t_{n-1}, (t_n, \dots, t_{2n-2}, (t_{2n-1}, \dots, t_{(3n-2)})))$$

and

$$F_2^+(t_1, \dots, t_{3n-2}) = (t_1, t_2, \dots, t_{n-2}, (t_{n-1}, \dots, t_{2n-2}), (t_{2n-1}, \dots, t_{(3n-2)}))$$

Their symmetric sums by all parameters except t_1 are defined by

$$F_1^{\{2\}}(t_1, \dots, t_{3n-2}) = \frac{1}{(n-1)!(n-2)!n!} \omega(t_{\{2, t_3, \dots, t_n, \omega(t_{\{1\}}, t_{n+1}, \dots, t_{2n-2}, \omega(t_{2n-1}, \dots, t_{(3n-2)}\})\})),$$

$$F_2^{\{2\}}(t_1, \dots, t_{3n-2}) = \frac{1}{(n-2)!(n-1)!n!} \omega(t_{\{2, t_3, \dots, t_{n-1}, \omega(t_{\{1\}}, t_n, \dots, t_{2n-2}), \omega(t_{2n-1}, \dots, t_{(3n-2)}\})\})).$$

Upper index s in $F_l^{\{s\}}$ corresponds to the place of ω , where t_1 is and lower index l corresponds to n -bracketing types of ω -degree 3. We have,

$$F_1^{\{2\}}(t_1, \dots, t_{3n-2}) = \sum_{\sigma \in \text{Sym}_{n-1, n-1, n}, \sigma(n)=1} (t_{\sigma(1)}, \dots, t_{\sigma(n-1)}, (t_1, t_{\sigma(n+1)}, \dots, t_{\sigma(2n-2)}, (t_{\sigma(2n-1)}, \dots, t_{\sigma(3n-2)}))),$$

$$F_2^{\{2\}}(t_1, \dots, t_{3n-2}) = \sum_{\sigma \in \text{Sym}_{n-2, n, n}, \sigma(n-1)=1} (t_{\sigma(1)}, \dots, t_{\sigma(n-2)}, (t_1, t_{\sigma(n)}, \dots, t_{\sigma(2n-2)}), (t_{\sigma(2n-1)}, \dots, t_{\sigma(3n-2)}))).$$

Let

$$f_\lambda^{\{2\}} = F_1^{\{2\}} + \lambda F_2^{\{2\}}.$$

Let A be n -algebra with n -multiplication (a_1, \dots, a_n) . Denote by $[A]$ an algebra with vector space A and n -multiplication

$$[a_1, \dots, a_n] = (a_{[1}, \dots, a_n]) = \sum_{\sigma \in \text{Sym}_n} \text{sign } \sigma (a_{\sigma(1)}, \dots, a_{\sigma(n)})$$

(Lie n -commutator). Similarly, denote by $\{A\}$ an algebra with vector space A and n -multiplication

$$\{a_1, \dots, a_n\} = (a_{\{1}, \dots, a_n\}) = \sum_{\sigma \in \text{Sym}_n} (a_{\sigma(1)}, \dots, a_{\sigma(n)})$$

(Jordan n -commutator).

Recall that A is called *total associative* [3] if

$$(a_1, \dots, a_i, (a_{i+1}, a_{i+2}, \dots, a_{i+n}), a_{i+n+1}, a_{i+n+2}, \dots, a_{2n-1}) = (a_1, \dots, a_i, a_{i+1}, (a_{i+2}, \dots, a_{i+n}, a_{i+n+1}), a_{i+n+2}, \dots, a_{2n-1}),$$

for any $1 \leq i \leq n-2$. Any associative algebra A under n -multiplication $(a_1, \dots, a_n) \mapsto a_1 \cdots a_n$ became total associative.

The following skew-symmetric ω -degree 2 polynomial is called *homotopical n -Lie*

$$\text{homot}(t_1, \dots, t_{2n-1}) = \sum_{\sigma \in S_{n-1, n}} \text{sign } \sigma \omega(t_{\sigma(1)}, \dots, t_{\sigma(n-1)}, \omega(t_{\sigma(n)}, \dots, t_{\omega(2n-1)})).$$

An n -ary algebra (A, ω) is called *homotopical n -Lie*, if it satisfies the identity $\text{homot} = 0$ [4].

Theorem 1.1. *Let A be total associative n -algebra. If n is even or if $\text{char } K = p > 0$ and n is divisible by p , then its n -commutators algebra $[A]$ is homotopical n -Lie.*

Theorem 1.2. *Let A be total associative algebra. Then its Lie n -commutators algebra $[A]$ satisfies the identity $f_{-1}^{[2]} = 0$.*

Theorem 1.3. *Let A be total associative algebra. Then its Jordan n -commutators algebra $\{A\}$ satisfies the identity $f_{-1}^{\{2\}} = 0$.*

Remarks. The fact that 3-commutators algebra $[A]$ has no identity of ω -degree 2 was noticed by A.G. Kurosh in [5]. The identity $f_{-1}^{[2]} = 0$ holds for any n -commutators algebra, but this identity in general is not minimal. If n is even or if the characteristic of main field is $p > 0$ and $n \equiv 0 \pmod{p}$, then one can find for $[A]$ the identity of ω -degree 2, for example, $\text{homot} = 0$. We think that $\text{homot} = 0$ for even n and $f_{-1}^{[2]} = 0$ for odd n are minimal identities that hold for any Lie n -commutator algebras $[A]$, if $\text{char } K = 0$. We think also that $f_{-1}^{\{2\}=0}$ is minimal identity that hold for any Jordan n -commutators algebra $\{A\}$.

The case $n = 3$ was considered by M. R. Bremner [1], [2]. He proved that $f_{-1}^{[2]} = 0$ and $f_{-1}^{\{2\}} = 0$ are identities for Lie and Jordan 3-commutators of total associative algebras and he established the minimality of these identities.

In case of $n = 2$ the polynomial $f_{-1}^{\{2\}}$ coincides with usual Jordan polynomial.

2. Proof of Theorem 1.1

Let A be a free total associative n algebra with n -multiplication ω and $[\omega]$ be its n -commutator,

$$[\omega](t_1, \dots, t_n) = \sum_{\sigma \in \text{Sym}_n} \text{sign } \sigma \omega(t_{\sigma(1)}, \dots, t_{\sigma(n)})$$

We have to prove that $X = 0$, where

$$X = X(t_1, \dots, t_{2n-1}) = \sum_{\sigma \in S_{n-1, n}} \text{sign } \sigma [\omega](t_{\sigma(1)}, \dots, t_{\sigma(n-1)}, [\omega](t_{\sigma(n)}, \dots, t_{\sigma(2n-1)})).$$

Expand n -commutators $[\omega]$ in terms of associative n -multiplication ω . We see that X is a sum of elements of a form

$$\pm \omega(t_{i_1}, \dots, t_{i_s}, \omega(t_{i_{s+1}}, \dots, t_{i_{s+n}}), t_{i_{s+n+1}}, \dots, t_{2n-1}).$$

Since A is total associative, this sum is reduced to a sum of elements of a form

$$\pm \omega(t_{j_1}, \dots, t_{j_{n-1}}, \omega(t_{j_n}, \dots, t_{j_{2n-1}})).$$

Let $\mu \in K$ be the coefficient of X at $\omega(t_1, \dots, t_{n-1}, \omega(t_n, \dots, t_{2n-1}))$. Since $X(t_1, \dots, t_{2n-1})$ is skew symmetric by all arguments t_1, \dots, t_{2n-1} to prove $X = 0$ it is enough to establish that $\mu = 0$.

Note that the element $Q := \omega(t_1, \dots, t_{n-1}, \omega(t_n, \dots, t_{2n-1}))$ may enter with non-zero coefficient only in summands of X of a form

$$\begin{aligned} R_{n-1} &:= [\omega](t_1, \dots, t_{n-1}, [\omega](t_n, \dots, t_{2n-1})), \\ R_{n-2} &:= [\omega](t_1, \dots, t_{n-2}, t_{2n-1}, [\omega](t_{n-1}, \dots, t_{2n-2})), \quad \dots, \\ R_0 &:= [\omega](t_{n+1}, \dots, t_{2n-1}, [\omega](t_1, \dots, t_n)). \end{aligned}$$

The element

$$R_i = [\omega](t_1, \dots, t_i, t_{n+i+1}, \dots, t_{2n-1}, [\omega](t_{i+1}, \dots, t_{i+n})), \quad 0 \leq i \leq n-1,$$

enter to X with coefficient that is equal to signature of the permutation

$$\gamma_i = \begin{pmatrix} 1 & \cdots & i & i+1 & \cdots & n-1 & n & \cdots & 2n-1 \\ 1 & \cdots & i & n+i+1 & \cdots & 2n-1 & i+1 & \cdots & i+n \end{pmatrix} \in S_{n-1,n}$$

We have

$$\text{sign } \gamma_i = (-1)^{(n-i-1)n}.$$

To obtain a component Q from R_i we have to permute the part $\omega(t_{i+1}, \dots, t_{i+n})$ of R_i $(n-i-1)$ times,

$$\begin{aligned} & [\omega](t_1, \dots, t_i, t_{n+i+1}, \dots, t_{2n-1}, [\omega](t_{i+1}, \dots, t_{i+n})) \rightsquigarrow \cdots \\ & \rightsquigarrow (-1)^{n-1-i} \omega(t_1, \dots, t_i, \omega(t_{i+1}, \dots, t_{i+n}), t_{i+n+1}, \dots, t_{2n-1}). \end{aligned}$$

Therefore

$$\mu = \sum_{i=0}^{n-1} \text{sign } \gamma_i (-1)^{n-i-1} = \sum_{i=0}^{n-1} (-1)^{(n-1-i)(n+1)}.$$

Note that

$$\mu = \begin{cases} 0 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

Hence, $X = 0$, if n even or $\text{char } K = p > 0$, n is odd and n is divisible by p .

3. Proof of Theorem 1.2

Note that

$$F_1^{[2]}(t_1, \dots, t_{3n-2}) = \sum_{\sigma \in \text{Sym}_{n-1, n-1, n}, \sigma(n)=1}$$

$$\text{sign } \sigma [\omega](t_{\sigma(1)}, \dots, t_{\sigma(n-1)}, [\omega](t_1, t_{\sigma(n+1)}, \dots, t_{\sigma(2n-2)}, [\omega](t_{\sigma(2n-1)}, \dots, t_{\sigma(3n-2)}))),$$

$$F_2^{[2]}(t_1, \dots, t_{3n-2}) = \sum_{\sigma \in \text{Sym}_{n-2, n, n}, \sigma(n-1)=1}$$

$$\text{sign } \sigma [\omega](t_{\sigma(1)}, \dots, t_{\sigma(n-2)}, [\omega](t_1, t_{\sigma(n)}, \dots, t_{\sigma(2n-2)}, [\omega](t_{\sigma(2n-1)}, \dots, t_{\sigma(3n-2)}))).$$

For any permutation $i_1 i_2 \dots i_{3n-2} \in \text{Sym}_{3n-2}$ set

$$e(i_1 \dots i_{3n-2}) := \omega(t_{i_1}, \dots, t_{i_{n-1}}, \omega(t_{i_n}, \dots, t_{i_{2n-2}}, \omega(t_{i_{2n-1}}, \dots, t_{i_{3n-2}})))$$

and

$$[e](i_1 \dots i_{3n-2}) := [\omega](t_{i_1}, \dots, t_{i_{n-1}}, [\omega](t_{i_n}, \dots, t_{i_{2n-2}}, [\omega](t_{i_{2n-1}}, \dots, t_{i_{3n-2}}))).$$

For any $1 \leq i \leq 3n-2$ let

$$e_i = e(2, \dots, i, 1, i+1, \dots, 3n-2)$$

The index i corresponds the place where is 1. For example, $e_1 = e(1, 2, \dots, 3n-2)$, $e_2 = e(2, 1, 3, \dots, 3n-2)$, $e_{3n-2} = e(2, \dots, 3n-2, 1)$.

Since A is total associative, for any $s = 1, 2$, the element $F_s^{[2]}(t_1, \dots, t_{3n-2})$ can be presented as a sum of elements $e(i_1, \dots, i_{3n-2})$, where $i_1 \dots i_{3n-2}$ is a permutation of the set $[3n-2] = \{1, 2, \dots, 3n-2\}$.

Let $\mu_{s,i}$ be a coefficient of $F_s^{[2]}(t_1, \dots, t_{3n-2})$ at $e_i = e(2, \dots, i, 1, i+1, \dots, 3n-2)$, where $1 \leq i \leq 3n-2$. Since $F_s^{[2]}(t_1, t_2, \dots, t_{3n-2})$ is skew-symmetric by all variables except t_1 , the element $F_s^{[2]}(t_1, t_2, \dots, t_{3n-2})$ is uniquely defined by

coefficient $\mu_{s,i}$, where $1 \leq i \leq 3n-2$. Then the condition that $f_{-1}^{[2]} = 0$ is identity on $[A]$ is equivalent to the following relations

$$(1) \quad \mu_{1,i} = \mu_{2,i}, \quad 1 \leq i \leq 3n-2.$$

We will establish the following common values for $\mu_{1,i}$ and $\mu_{2,i}$.

Let for even n

$$\mu_i = \begin{cases} (-1)^{(i+1)} \lfloor \frac{i+1}{2} \rfloor & \text{if } i \leq n \\ (-1)^{i+1} (\frac{n}{2} + 2 \lfloor \frac{n-i-1}{2} \rfloor) & \text{if } n+1 \leq i \leq 2n-2 \\ (-1)^i \lfloor \frac{3n-i}{2} \rfloor & \text{if } 2n-1 \leq i \leq 3n-2 \end{cases}$$

and for odd n

$$\mu_i = \begin{cases} (-1)^{(i+1)} \frac{(2n-i-1)i}{2} & \text{if } i \leq n \\ (-1)^{(i+1)} (\frac{n(n-1)}{2} + (i-n)(-2n+i+1)) & \text{if } n+1 \leq i \leq 2n-2 \\ (-1)^i \frac{(3n-i-1)(n-i)}{2} & \text{if } 2n-1 \leq i \leq 3n-2 \end{cases}$$

Note that

$$\mu_i = \mu_{3n-1-i}, \quad 1 \leq i \leq 3n-2.$$

Let $[i, j] = \{s \in \mathbf{Z} | i \leq s \leq j\}$ be segment with endpoints i, j and $[i, j) = \{s \in \mathbf{Z} | i \leq s < j\}$, $(i, j] = \{s \in \mathbf{Z} | i < s \leq j\}$, $(i, j) = \{s \in \mathbf{Z} | i < s < j\}$ be semi-segments. Note that semi-segment $[i, j)$ has endpoints i and $j-1$ and similarly, endpoints of $(i, j]$ is $i+1$ and j . Number of elements of (semi)-segment is called length. For example, $|[i, j]| = j-i$ and $|[i, j)| = j-1-i$, if $j > i$. Say that $[i_1, j_1] \subseteq [i, j]$ is subsegment if $i \leq i_1 < j_1 \leq j$.

Lemma 3.1. *Let $\mu_{1,i}$ be the coefficient at e_i of the element $F_1^{[2]}(t_1, \dots, t_{3n-2})$. Then*

$$\mu_{1,i} = \mu_i,$$

for any $1 \leq i \leq 3n-2$.

Proof. Consider in the segment $P_1 = [2, 3n-2] = \{2, \dots, 3n-2\}$ chain with two subsegments

$$P_3 \subset P_2 \subset P_1, \quad |P_1| = 3n-3, |P_2| = 2n-2, |P_3| = n$$

Denote endpoints of P_1, P_2, P_3 as A_1, B_1 , A_2, B_2 and A_3, B_3 . Then

$$P_1 = [2, 3n-2], P_2 = [p+1, p+2n-2], P_3 = [q, q+n-1]$$

for some $1 \leq p < q \leq 2n-1$ and the points $A_1, A_2, A_3, B_3, B_2, B_1$ on \mathbf{R} has coordinates $2, p+1, q, q+n-1, p+2n-2, 3n-2$. Note that

$$(2) \quad 1 \leq p \leq n, \quad p < q \leq 2n-1, \quad q \leq p+n-1$$

Then

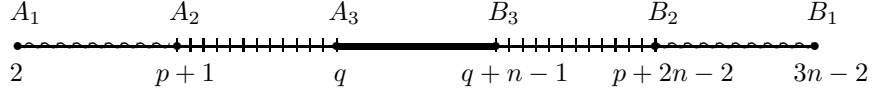
$$P_1 = [A_1, A_2) \cup P_2 \cup (B_2, B_1],$$

$$P_2 = [A_2, A_3) \cup P_3 \cup (B_3, B_2].$$

Let us introduce the following subsets of increasing integers

$$\begin{aligned} X_1 &= [A_1, A_2) \cup (B_2, B_1], \\ X_2 &= \{1\} \cup [A_2, A_3) \cup (B_3, B_2], \\ X_3 &= P_3 = [A_3, B_3]. \end{aligned}$$

In the following picture parts of X_1, X_2, X_3 are marked equally.



So, for any such chain $P_1 \supset P_2 \supset P_3$ one corresponds a sequence of elements $X_1 X_2 X_3$ where in each part X_i elements are written in increasing order and X_2 begins by 1. In other words, any chain $P_1 \supset P_2 \supset P_3$ defines in a unique way an element $[e](X_1 X_2 X_3)$. More exactly,

$$[e](X_1 X_2 X_3) =$$

$$[\omega](t_2, \dots, t_p, t_{p+2n-1}, \dots, t_{3n-2}, [\omega](t_1, t_{p+1}, \dots, t_{q-1}, t_{q+n}, \dots, t_{p+2n-2}, [\omega](t_q, \dots, t_{q+n-1}))).$$

Signature of the permutation

$$X_1 X_2 X_3 = 2 \dots p p+2n-1 \dots 3n-2 1 p+1 \dots q-1 q+n \dots p+2n-2 q \dots q+n-1$$

is equal to

$$(-1)^{(p-1)+(n-p)(2n-1)+(n-q+p-1)n}.$$

So,

$$(3) \quad \operatorname{sgn} X_1 X_2 X_3 = (-1)^{(p+1-q)n+1}$$

For $1 \leq i \leq 3n-2$ and $1 \leq p \leq n$, $0 < q-p \leq n-1$, denote by $\mu_{1,i}^{(p,q)}$, the coefficient at e_i of the element

$$[\omega](t_2, \dots, t_p, t_{p+2n-1}, \dots, t_{3n-2}, [\omega](t_1, t_{p+1}, \dots, t_{q-1}, t_{q+n}, \dots, t_{p+2n-2}, [\omega](t_q, \dots, t_{q+n-1}))).$$

In case of $p=1$, by $\mu_{1,i}^{(p,q)}$ we understand the coefficient at e_i of the element

$$[\omega](t_2, \dots, t_{3n-2}, [\omega](t_1, t_2, \dots, t_{q-1}, t_{q+n}, \dots, t_{p+2n-2}, [\omega](t_q, \dots, t_{q+n-1}))).$$

In case of $p=n$, by $\mu_{1,i}^{(p,q)}$ we mean the coefficient at e_i of the element

$$[\omega](t_2, \dots, t_n, [\omega](t_1, t_{n+1}, \dots, t_{q-1}, t_{q+n}, \dots, t_{p+2n-2}, [\omega](t_q, \dots, t_{q+n-1}))).$$

For any $1 \leq i \leq 3n-2$ denote by $\mu_{1,i}^{(0,q)}$ the coefficient at e_i of the element

$$\omega(t_2, \dots, t_{3n-2}, \omega(t_1, t_{n+2}, \dots, t_{2n-1}, \omega(t_2, \dots, t_{n+1}))).$$

Then $\mu_{1,i}^{(0,q)} = 0$, if $i \leq n$ or $i \geq 2n-1$.

Below we use the following notation $Y \rightsquigarrow Z$ that means that Z is obtained from Y by using skew-symmetry property of $[\omega]$

We have,

$$[\omega](t_2, \dots, t_p, t_{p+2n-1}, \dots, t_{3n-2}, [\omega](t_1, t_{p+1}, \dots, t_{q-1}, t_{q+n}, \dots, t_{p+2n-2}, \omega(t_q, \dots, t_{q+n-1}))) \rightsquigarrow$$

$$(-1)^{(p-q+n-1)} [\omega](t_2, \dots, t_p, t_{p+2n-1}, \dots, t_{3n-2}, [\omega](t_1, t_{p+1}, \dots, t_{q-1}, \omega(t_q, \dots, t_{q+n-1}), t_{q+n}, \dots, t_{p+2n-2})) \rightsquigarrow$$

$$(-1)^{(q-1)}[\omega](t_2, \dots, t_p, [\omega](t_1, t_{p+1}, \dots, t_{q-1}, \omega(t_q, \dots, t_{q+n-1}), t_{q+n} \dots, t_{p+2n-2}), t_{p+2n-1}, \dots, t_{3n-2}) \rightsquigarrow$$

$$(-1)^{(q-1)}\omega(t_2, \dots, t_p, [\omega](t_1, t_{p+1}, \dots, t_{q-1}, \omega(t_q, \dots, t_{q+n-1}), t_{q+n} \dots, t_{p+2n-2}), t_{p+2n-1}, \dots, t_{3n-2})$$

Now expand $[\omega]$ in $[\omega](t_1, t_{p+1}, \dots, t_{q-1}, \omega(t_q, \dots, t_{q+n-1}), t_{q+n} \dots, t_{p+2n-2})$. Then t_1 might be in i -th place only in the following cases

$$(4) \quad \omega(t_1, t_{p+1}, \dots, t_{q-1}, \omega(t_q, \dots, t_{q+n-1}), t_{q+n} \dots, t_{p+2n-2}), \quad i = p,$$

$$(5) \quad (-1)^{i-p}\omega(t_{p+1}, \dots, t_i, t_1, \dots, t_{q-1}, \omega(t_q, \dots, t_{q+n-1}), t_{q+n} \dots, t_{p+2n-2}), \quad p+1 \leq i \leq q-1,$$

$$(6) \quad (-1)^{n-1-p+i}\omega(t_{p+1}, \dots, t_{q-1}, \omega(t_q, \dots, t_{q+n-1}), t_{q+n} \dots, t_i, t_1, \dots, t_{p+2n-2}), \quad q+n-1 \leq i \leq p+2n-2$$

Note that $\mu_{1,i}^{(p,q)} = 0$, if $i \notin [A_2, A_3] \cup (B_3, B_2]$. Therefore, by (4), (5) and (6),

$$(7) \quad \mu_{1,i}^{(p,q)} = \begin{cases} 0 & \text{if } i < p \text{ or } q \leq i \leq q+n-1 \text{ or } p+2n-2 \leq i \leq 3n-2 \\ (-1)^{i+1+p-q} & \text{if } p \leq i \leq q-1 \\ (-1)^{n+p+i-q} & \text{if } q+n-1 \leq i \leq p+2n-2 \end{cases}$$

Note also $1 \leq p \leq n, p < q \leq p+n-1$. Hence $q \leq 2n-1$.

The element e_i , where $1 \leq i \leq 3n-2$, may appear in expanding of

$$[\omega](t_2, \dots, t_p, t_{p+2n-1}, \dots, t_{3n-2}, [\omega](t_1, t_{p+1}, \dots, t_{q-1}, t_{q+n} \dots, t_{p+2n-2}, [\omega](t_q, \dots, t_{q+n-1})))$$

with the coefficient

$$(8) \quad \mu_{1,i} = \sum_{p,q} \mu_{1,i}^{(p,q)}.$$

Let $i \leq n$. Then the case $q+n-1 \leq i, 1 \leq p < q$ is impossible. Therefore, by (1), (3), (7),

$$\mu_{1,i} = \sum_{p=1}^i \sum_{q=i+1}^{p+n-1} \mu_{1,i}^{(p,q)} = \sum_{p=1}^i \sum_{q=i+1}^{p+n-1} (-1)^{(p+1-q)n+1} (-1)^{p-q+i+1} = \sum_{p=1}^i \sum_{q=i+1}^{p+n-1} (-1)^{p-q+i+pn+qn+n}.$$

So, for even n ,

$$\mu_{1,i} = \sum_{p=1}^i \sum_{q=i+1}^{p+n-1} (-1)^{p-q+i} = (-1)^i \sum_{p=1}^i \sum_{q=i+1}^{p+n-1} (-1)^{p-q} = (-1)^{i+1} \lfloor \frac{i+1}{2} \rfloor.$$

For odd n

$$\mu_{1,i} = \sum_{p=1}^i \sum_{q=i+1}^{p+n-1} (-1)^{i+1} = (-1)^i \sum_{p=1}^i \sum_{q=i+1}^{p+n-1} 1 = (-1)^{i+1} \frac{(2n-i-1)i}{2}.$$

Consider the case $n + 1 \leq i \leq 2n - 2$. By (1), (3), (7),

$$\mu_{1,i} = \sum_{p=1}^n \sum_{q=i+1}^{p+n-1} (-1)^{(p+1-q)n+1} (-1)^{i+1+p-q} + \sum_{p=0}^n \sum_{q=p+1}^{i-n} (-1)^{(p+1-q)n+1} (-1)^{n+p+i-q}$$

So, if n is even, then

$$\begin{aligned} \mu_{1,i} &= \sum_{p=1}^n \sum_{q=i+1}^{p+n-1} (-1)^{i+p-q} - \sum_{p=0}^n \sum_{q=p+1}^{i-n} (-1)^{i-q+p} = \\ &= (-1)^i \left(\sum_{p=1}^n \sum_{q=i+1}^{p+n-1} (-1)^{p-q} - \sum_{p=0}^n \sum_{q=p+1}^{i-n} (-1)^{q-p} \right) = \\ &= (-1)^{i+1} \left(\frac{n}{2} + 2 \left\lfloor \frac{n-i-1}{2} \right\rfloor \right). \end{aligned}$$

If n is odd, then

$$\begin{aligned} \mu_{1,i} &= \sum_{p=1}^n \sum_{q=i+1}^{p+n-1} (-1)^{i+1} + \sum_{p=0}^n \sum_{q=p+1}^{i-n} (-1)^{i-1} = \\ &= (-1)^{i+1} \left(\frac{n(n-1)}{2} + (i-n)(-2n+i+1) \right). \end{aligned}$$

Consider the case $2n - 1 \leq i \leq 3n - 2$. Then all cases except $q + n - 1 \leq i \leq p + 2n - 2$ are not possible. Therefore, $i - 2n + 2 \leq p < q \leq i - n + 1$, and

$$\mu_{1,i} = \sum_{p=i-2n+2}^n \sum_{q=p+1}^{i-n+1} \mu_{1,i}^{(p,q)} = \sum_{p=i-2n+2}^n \sum_{q=p+1}^{i-n+1} (-1)^{(p+1-q)n+1} (-1)^{n+p+i-q}$$

Hence, for even n

$$\mu_{1,i} = \sum_{p=i-2n+2}^n \sum_{q=p+1}^{i-n+1} (-1)^{i+p-q+1} = (-1)^{i+1} \sum_{p=i-2n+2}^n \sum_{q=p+1}^{i-n+1} (-1)^{p-q} = (-1)^i \left\lfloor \frac{3n-i}{2} \right\rfloor.$$

and for odd n

$$\mu_{1,i} = \sum_{p=i-2n+2}^n \sum_{q=p+1}^{i-n+1} (-1)^{i+1} = (-1)^{i+1} \sum_{p=i-2n+2}^n \sum_{q=p+1}^{i-n+1} 1 = (-1)^i \frac{(3n-i-1)(n-i)}{2}.$$

Lemma 3.1 is proved.

Lemma 3.2. Let $\mu_{2,i}$ be the coefficient at e_i of the element $F_2^{[2]}(t_1, \dots, t_{3n-2})$. Then

$$\mu_{2,i} = \mu_i,$$

for any $1 \leq i \leq 3n - 2$.

Proof. Consider in the segment $P_1 = [2, 3n - 2] = \{2, \dots, 3n - 2\}$ two non-intersecting subsegments of length $n - 1$ and n

$$P_1 \supset P_2, \quad P_1 \supset P_3, \quad |P_1| = 3n - 3, |P_2| = n - 1, |P_3| = n.$$

Denote endpoints of P_1, P_2, P_3 as A_1, B_1, A_2, B_2 and A_3, B_3 . Then

$$P_1 = [2, 3n - 2], \quad P_2 = [p + 1, p + n - 1], \quad P_3 = [q, q + n - 1]$$

for some $1 \leq p \leq 2n - 1$ and $2 \leq q \leq 2n - 1$.

Note that

$$\begin{aligned} q &\geq p + n \text{ if } p < q \\ p &\geq q + n - 1 \text{ if } p > q \end{aligned}$$

Then

$$P_1 = [A_1, B_1], \quad P_2 = [A_2, B_2], \quad P_3 = [A_3, B_3]$$

Let us introduce the following subsets of increasing integers

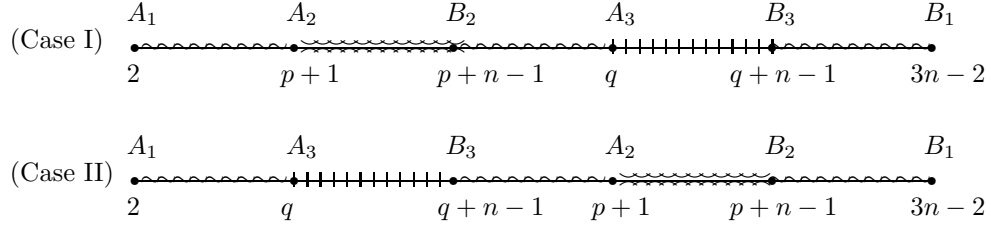
$$X_1 = [A_1, A_2] \cup (B_2, A_3) \cup (B_3, B_1) \quad (\text{Case I}),$$

$$X_1 = [A_1, A_3] \cup (B_3, A_2) \cup (B_2, B_1) \quad (\text{Case II}),$$

$$X_2 = \{1\} \cup [A_2, B_2],$$

$$X_3 = P_3 = [A_3, B_3].$$

In the following picture parts of X_1, X_2, X_3 are marked equally.



Note that $|X_1| = n - 2, |X_2| = n, |X_3| = n$.

In Case I we have an element

$$[e](X_1, X_2, X_3) =$$

$$[\omega](t_2, \dots, t_p, t_{p+n-1}, \dots, t_{q-1}, t_{q+n}, \dots, t_{3n-2}, [\omega](t_1, t_{p+1}, \dots, t_{p+n-1}), [\omega](t_q, \dots, t_{q+n-1}))$$

with signature

$$(-1)^{|X_1| + |(B_2, A_3)| + |[A_2, B_2]| + |(B_3, B_1)| + |[A_2, B_2]| + |(B_3, B_1)| + |[A_3, B_3]|}.$$

Note that

$$|X_1| = n - 2 \equiv n \pmod{2},$$

$$|(B_2, A_3)| = |[B_2, A_3]| - 2 \equiv |[B_2, A_3]| = q - p - n \pmod{2},$$

$$|(B_3, B_1)| = |[B_3, B_1]| - 1 = 3n - 2 - q + n - 1 \equiv q + 1 \pmod{2},$$

$$|[A_2, B_2]| \equiv n - 1 \pmod{2},$$

$$|[A_3, B_3]| \equiv n \pmod{2}.$$

Therefore, in Case I, $q \geq p + n$, and

$$\text{sign } X_1 X_2 X_3 = (-1)^{n + (q-p-n)(n-1) + (q+1)(n-1) + (q+1)n} = (-1)^{(q-p+1)n + p+1}.$$

In Case II $p \geq q + n - 1$ and we have an element

$$[e](X_1, X_2, X_3) =$$

$$[\omega](t_2, \dots, t_{q-1}, t_{q+n}, \dots, t_p, t_{p+n}, \dots, t_{3n-2}, [\omega](t_1, t_{p+1}, \dots, t_{p+n-1}), [\omega](t_q, \dots, t_{q+n-1}))$$

with signature

$$(-1)^{|X_1| + |(B_3, A_2)| + |[A_3, B_3]| + |(B_2, B_1)| + |[A_3, B_3]| + |(B_2, B_1)| + |[A_2, B_2]| + |[A_2, B_2]| + |[A_3, B_3]|}.$$

Since

$$\begin{aligned} |X_1| &\equiv n \pmod{2}, \quad |[A_3, B_3]| = n, \quad |[A_2, B_2]| \equiv n-1 \pmod{2}, \\ |(B_3, A_2)| &\equiv |[B_3, A_2]| \equiv p-q-n+1 \pmod{2}, \\ |(B_2, B_1)| &= |[B_2, B_1]| - 1 \equiv 3n-2-p-n+1 \equiv p-1 \pmod{2}, \end{aligned}$$

we have

$$\text{sign } X_1 X_2 X_3 = (-1)^{n+(p-q-n+1)n+(p-1)n+(p-1)(n-1)+(n-1)n} = (-1)^{(p-q)n+p-1+n}.$$

So,

$$(9) \quad \text{sign } X_1 X_2 X_3 = \begin{cases} (-1)^{(q+p+1)n+p+1} & \text{Case I, } q \geq p+n \\ (-1)^{(p+q+1)n+p+1} & \text{Case II, } p \geq q+n \end{cases}$$

For $1 \leq i \leq 3n-2$ denote by $\mu_{2,i}^{(p,q)}$, the coefficient at $e_i = e(2, \dots, i, 1, i+1, \dots, 3n-2)$ of the element

$$[\omega](t_2, \dots, t_p, t_{p+n-1}, \dots, t_{q-1}, t_{q+n}, \dots, t_{3n-2}, [\omega](t_1, t_{p+1}, \dots, t_{p+n-1}), [\omega](t_q, \dots, t_{q+n-1})),$$

in Case I or of the element

$$[\omega](t_2, \dots, t_{q-1}, t_{q+n}, \dots, t_p, t_{p+n-1}, \dots, t_{3n-2}, [\omega](t_1, t_{p+1}, \dots, t_{p+n-1}), [\omega](t_q, \dots, t_{q+n-1})),$$

in Case II.

To calculate $\mu_{2,i}^{(p,q)} e_i$ we have to do the following permutations.

Case I. $p+n \leq q$, $p \leq i \leq p+n-1$.

$$[\omega](t_2, \dots, t_p, t_{p+n}, \dots, t_{q-1}, t_{q+n}, \dots, t_{3n-2}, [\omega](t_1, t_{p+1}, \dots, t_{p+n-1}), [\omega](t_q, \dots, t_{q+n-1})) \rightsquigarrow$$

$$(-1)^{p+n+1} [\omega](t_2, \dots, t_p, [\omega](t_1, t_{p+1}, \dots, t_{p+n-1}), t_{p+n}, \dots, t_{q-1}, t_{q+n}, \dots, t_{3n-2}, \omega(t_q, \dots, t_{q+n-1})) \rightsquigarrow$$

$$(-1)^{p+q+n} [\omega](t_2, \dots, t_p, [\omega](t_1, t_{p+1}, \dots, t_{p+n-1}), t_{p+n}, \dots, t_{q-1}, \omega(t_q, \dots, t_{q+n-1}), t_{q+n}, \dots, t_{3n-2}) \rightsquigarrow$$

$$(-1)^{i+q+n} \omega(t_2, \dots, t_p, \omega(t_{p+1}, \dots, t_i, t_1, t_{i+1}, \dots, t_{p+n-1}), t_{p+n}, \dots, t_{q-1},$$

$$\omega(t_q, \dots, t_{q+n-1}), t_{q+n}, \dots, t_{3n-2}) \rightsquigarrow$$

(by total associativity)

$$\rightsquigarrow (-1)^{i+q+n} e(2, \dots, i, 1, i+1, \dots, 3n-2) = (-1)^{i+q+n} e_i.$$

Case II, $q+n-1 \leq p$, $p \leq i \leq p+n-1$.

$$[\omega](t_2, \dots, t_{q-1}, t_{q+n}, \dots, t_p, t_{p+n}, \dots, t_{3n-2}, [\omega](t_1, t_{p+1}, \dots, t_{p+n-1}), [\omega](t_q, \dots, t_{q+n-1})) \rightsquigarrow$$

$$(-1)^{p+1} [\omega](t_2, \dots, t_{q-1}, t_{q+n}, \dots, t_p, [\omega](t_1, t_{p+1}, \dots, t_{p+n-1}), t_{p+n}, \dots, t_{3n-2}, \omega(t_q, \dots, t_{q+n-1})) \rightsquigarrow$$

$$(-1)^{p-q-n} [\omega](t_2, \dots, t_{q-1}, \omega(t_q, \dots, t_{q+n-1}), t_{q+n}, \dots, t_p, [\omega](t_1, t_{p+1}, \dots, t_{p+n-1}), t_{p+n}, \dots, t_{3n-2}) \rightsquigarrow$$

$$\begin{aligned}
& (-1)^{i-q-n} \omega(t_2, \dots, t_{q-1}, \omega(t_q, \dots, t_{q+n-1}), t_{q+n}, \dots, t_p, \omega(t_{p+1}, \dots, t_i, t_1, t_{i+1}, \dots, t_{p+n-1}), \\
& \qquad \qquad \qquad t_{p+n}, \dots, t_{3n-2}) \rightsquigarrow \\
& \text{(by total associativity)} \\
& \rightsquigarrow (-1)^{i-q-n} e(2, \dots, i, 1, i+1, \dots, 3n-2) = (-1)^{i-q-n} e_i.
\end{aligned}$$

Consider the case $i \leq n$. Then the Case II is impossible, P_2 is on the left of P_3 . We have

$$\mu_{2,i} = \sum_{p,q} \mu_{2,i}^{(p,q)} = \sum_{p=1}^i \sum_{q=p+n}^{2n-1} (-1)^{(q+p+1)n+p+1} (-1)^{i+q+n} = \sum_{p=1}^i \sum_{q=p+n}^{2n-1} (-1)^{(q+p+1)(n+1)+i+n}$$

So, if n is even,

$$\mu_{2,i} = \sum_{p=1}^i \sum_{q=p+n}^{2n-1} (-1)^{q+p+1+i} = (-1)^{i+1} \sum_{p=1}^i \sum_{q=p+n}^{2n-1} (-1)^{q-p} = (-1)^{i+1} \lfloor \frac{i+1}{2} \rfloor,$$

and if n is odd,

$$\mu_{2,i} = \sum_{p=1}^i \sum_{q=p+n}^{2n-1} (-1)^{i+1} = (-1)^{i+1} \frac{(2n-i-1)i}{2}.$$

Consider the case $n+1 \leq i \leq 2n-2$. Then

$$\begin{aligned}
\mu_{2,i} &= \sum_{p,q} \mu_{2,i}^{(p,q)} = \sum_{p=1}^i \sum_{q=p+n}^{2n-1} (-1)^{(q+p+1)n+p+1} (-1)^{i+q+n} + \sum_{p=i-n+1}^i \sum_{q=2}^{p-n+1} (-1)^{(q+p+1)n+p+1} (-1)^{i-q-n} = \\
& (-1)^i \left(\sum_{p=1}^i \sum_{q=p+n}^{2n-1} (-1)^{(q+p+1)(n+1)+n} + \sum_{p=i-n+1}^i \sum_{q=2}^{p-n+1} (-1)^{(q+p+1)(n+1)+n} \right)
\end{aligned}$$

So, if n is even, then

$$\mu_{2,i} = (-1)^i \left(\sum_{p=i-n+1}^i \sum_{q=p+n}^{2n-1} (-1)^{q+p+1} + \sum_{p=i-n+1}^i \sum_{q=2}^{p-n+1} (-1)^{q+p+1} \right) = (-1)^{i+1} \left(\frac{n}{2} + 2 \lfloor \frac{n-i-1}{2} \rfloor \right),$$

and if n is odd

$$\mu_{2,i} = (-1)^{i+1} \left(\sum_{p=i-n+1}^i \sum_{q=p+n}^{2n-1} 1 + \sum_{p=i-n+1}^i \sum_{q=2}^{p-n+1} 1 \right) = (-1)^{i+1} \left(\frac{n(n-1)}{2} + (i-n)(-2n+i+1) \right).$$

Consider the case $2n-1 \leq i \leq 3n-2$. Then Case I is impossible, and P_2 is on the right sight of P_3 . We have

$$\mu_{2,i} = \sum_{p,q} \mu_{2,i}^{(p,q)} = \sum_{p=i-n+1}^{2n-1} \sum_{q=2}^{p-n+1} (-1)^{(q+p+1)n+p+1} (-1)^{i-q-n} = \sum_{p=i-n+1}^{2n-1} \sum_{q=2}^{p-n+1} (-1)^{(q+p+1)(n+1)+i+n}$$

So, if n is even,

$$\mu_{2,i} = (-1)^{i+1} \sum_{p=i-n+1}^{2n-1} \sum_{q=2}^{p-n+1} (-1)^{q-p} = (-1)^i \lfloor \frac{3n-i}{2} \rfloor$$

and if n is odd,

$$\mu_{2,i} = (-1)^{i+1} \sum_{p=i-n+1}^{2n-1} \sum_{q=2}^{p-n+1} 1 = (-1)^i \frac{(3n-i-1)(n-i)}{2}.$$

Lemma 3.2 is proved completely.

Proof of Theorem 1.2. It follows from Lemmas 3.1 and 3.2.

4. Proof of Theorem 1.3

Repeats arguments of the proof of Theorem 1.2. Let

$$\mu_i^+ = \begin{cases} \frac{(2n-i-1)i}{2} & \text{if } i \leq n \\ \frac{n(n-1)}{2} + (i-n)(-2n+i+1) & \text{if } n+1 \leq i \leq 2n-2 \\ \frac{(3n-i-1)(i-n)}{2} & \text{if } 2n-1 \leq i \leq 3n-2 \end{cases}$$

Lemma 4.1. Let $\mu_{1,i}^+$ be the coefficient at e_i of the element $F_1^{\{2\}}(t_1, \dots, t_{3n-2})$. Then

$$\mu_{1,i}^+ = \mu_i^+,$$

for any $1 \leq i \leq 3n-2$.

Lemma 4.2. Let $\mu_{2,i}^+$ be the coefficient at e_i of the element $F_2^{\{2\}}(t_1, \dots, t_{3n-2})$. Then

$$\mu_{2,i}^+ = \mu_i^+,$$

for any $1 \leq i \leq 3n-2$.

Theorem 1.3 follows from Lemmas 4.1 and 4.2.

Remark. Note that $\mu_i = \mu_i^+(-1)^{i+1}$, $1 \leq i \leq 3n-2$, if n is odd. The generating function for μ_i^+ is a product of two polynomials,

$$G_n(x) = \sum_{i=1}^{3n-2} \mu_i^+ x^i = \left(\sum_{i=1}^n x^i \right) \left(\sum_{i=1}^{n-1} (n-i)x^{i-1} + i x^{i+n-1} \right).$$

or,

$$G_n(x) = \sum_{i=1}^{3n-2} \mu_i^+ x^i = \left(\sum_{i=1}^n x^i \right) (nx^{-1} + (x^n - 1)\partial) \left(\sum_{i=1}^{n-1} x^i \right).$$

If $n = 2k$ is even, then the generating function for μ_i is the following polynomial

$$Q_{2k}(x) = \sum_{i=1}^{6k-2} \mu_i x^i = (x-1)^2 x(x+1) \left(\sum_{i=1}^k x^{2i-2} \right)^3,$$

or,

$$Q_n(x) = \sum_{i=1}^{3n-2} \mu_i x^i = \frac{x(x-1)(x^n-1)^3}{(1-x^2)^2}.$$

Therefore, we can formulate the following more exact versions of Theorems 1.2, 1.3.

$$F_1^{[2]} = F_2^{[2]} = \sum_{i=1}^{3n-2} \mu_i [e_i],$$

$$F_1^{\{2\}} = F_2^{\{2\}} = \sum_{i=1}^{3n-2} \mu_i^+ \{e_i\},$$

where μ_i^+ are defined as coefficients of the polynomial $G_n(x)$ and μ_i are coefficients of the polynomial $-G_n(-x)$ for odd n and $Q_n(x)$ for even n .

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